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Where to place the future SLR satellite for the best GM, geocenter, C_{20}, and other gravity field parameters recovery?

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Kaula parameterization – expansion of the potential into trigonometric series

$$V_{nm} = -\frac{GM}{a}\left(\frac{a_e}{a}\right)^n \sum_{p=0}^{n} \sum_{q=-\infty}^{\infty} F_{nmp}(i) G_{npq}(e) P_{nm} \begin{pmatrix} \cos\Psi_{nmpq} \\ \sin\Psi_{nmpq} \end{pmatrix}_{n-m\ \mathrm{odd}}^{n-m\ \mathrm{even}}$$

$$\Psi_{nmpq} = (n-2p)\omega + (n-2p+q)M + m(\Omega - \theta - \lambda_{nm})$$

$F_{nmp}(i)$ – inclination function

$G_{npq}(e)$– eccentricity function

- $(n-2p+q) \neq 0$ short-periodic perturbations
- $(n-2p+q) = 0$ and $m = 0$ long-periodic perturbations (for odd-degree zonal harmonics)
- $m \neq 0$ m-daily perturbations
- $(n-2p+q) = 0$ and $m = 0$ and $(n-2p) = 0$ secular perturbations (for even-degree zonal harmonics)

Inclination function

The inclination function describes the relations between the inclination angle *i* and *n, m, p* parameters:

$$F_{n,m,p}(i) = \sum_{t=0}^{\min(p,k)} \left(\frac{(2n-2t)!}{t!(n-t)!(n-m-2t)! \cdot 2^{2n-2t}} \cdot (\sin \mathrm{i})^{n-m-2t} \cdot \sum_{s=0}^{m} \binom{m}{s} (\cos \mathrm{i})^{s} \cdot \sum_{c=0}^{5} \binom{n-m-2t+s}{c} \binom{m-s}{p-t-c} (-1)^{c-k} \right)$$

$$k = \lfloor \frac{n-m}{2} \rfloor;$$

Please note that the odd-degree gravity field parameters, e.g., C_{30} or C_{50} do not cause any secular orbit perturbations, because it is impossible to nullify the $(n-2p)$ term for odd-degree *n* with *p* being a natural number (including zero). Therefore, the odd-degree zonal coefficients can be derived either from short-periodic or long-periodic perturbations, whereas the even-degree zonal coefficients mostly benefit from the secular rates of Keplerian parameters (Ω and ω for non-circular orbits) with the contribution from other perturbation types.

Eccentricity function

$$G_{n,p,q}(e) = (-1)^{|q|} \cdot (1+\beta^2)^n \cdot \beta^{|q|} \cdot \sum_{k=0}^{10} \left(\sum_{r_p=0}^{h_p} \binom{2p_p - 2n}{h_p - r_p} \frac{(-1)^{r_p}}{r_p!} \left(\frac{(n - 2p_p + q_p)e}{2\beta} \right)^{r_p} \right) \cdot$$
$$\left(\sum_{r_q=0}^{h_q} \binom{-2p_p}{h_q - r_q} \frac{1}{r_q!} \left(\frac{(n - 2p_p + q_p)e}{2\beta} \right)^{r_q} \right) \cdot \beta^{2k}$$

Please note that for q=0 the term $\beta^{|q|}=1$ and then for small eccentricities the entire function $G_{npq}(e)=1$.

$$\beta = \frac{e}{1+\sqrt{1-e^2}},$$

$$p_p = \begin{cases} n-p & \text{if } p > \frac{n}{2} \\ p & \text{otherwise} \end{cases}, \quad q_p = \begin{cases} -q & \text{if } p > \frac{n}{2} \\ q & \text{otherwise} \end{cases}$$

$$h_p = \begin{cases} k & \text{if } q_p < 0 \\ k + q_p & \text{otherwise} \end{cases}, \quad h_q = \begin{cases} k - q_p & \text{if } q_p < 0 \\ k & \text{otherwise} \end{cases}$$

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Recovery of zonal even and odd harmonics based on Keplerian parameter perturbations

Example – C_{20}

Recovery of **even zonals** based on secular rates of the **ascending node**:

$$D_{20} = -\frac{\sqrt{\frac{GM}{r^3}}}{\sin(\mathrm{i})}\left(\frac{a_e}{r}\right)^2 \frac{G}{\sqrt{1-e^2}}\frac{dF}{di}$$

n=2, m=0, p=1, q=0

$$F = \frac{3\sin^2(i)}{4} - \frac{1}{2}$$

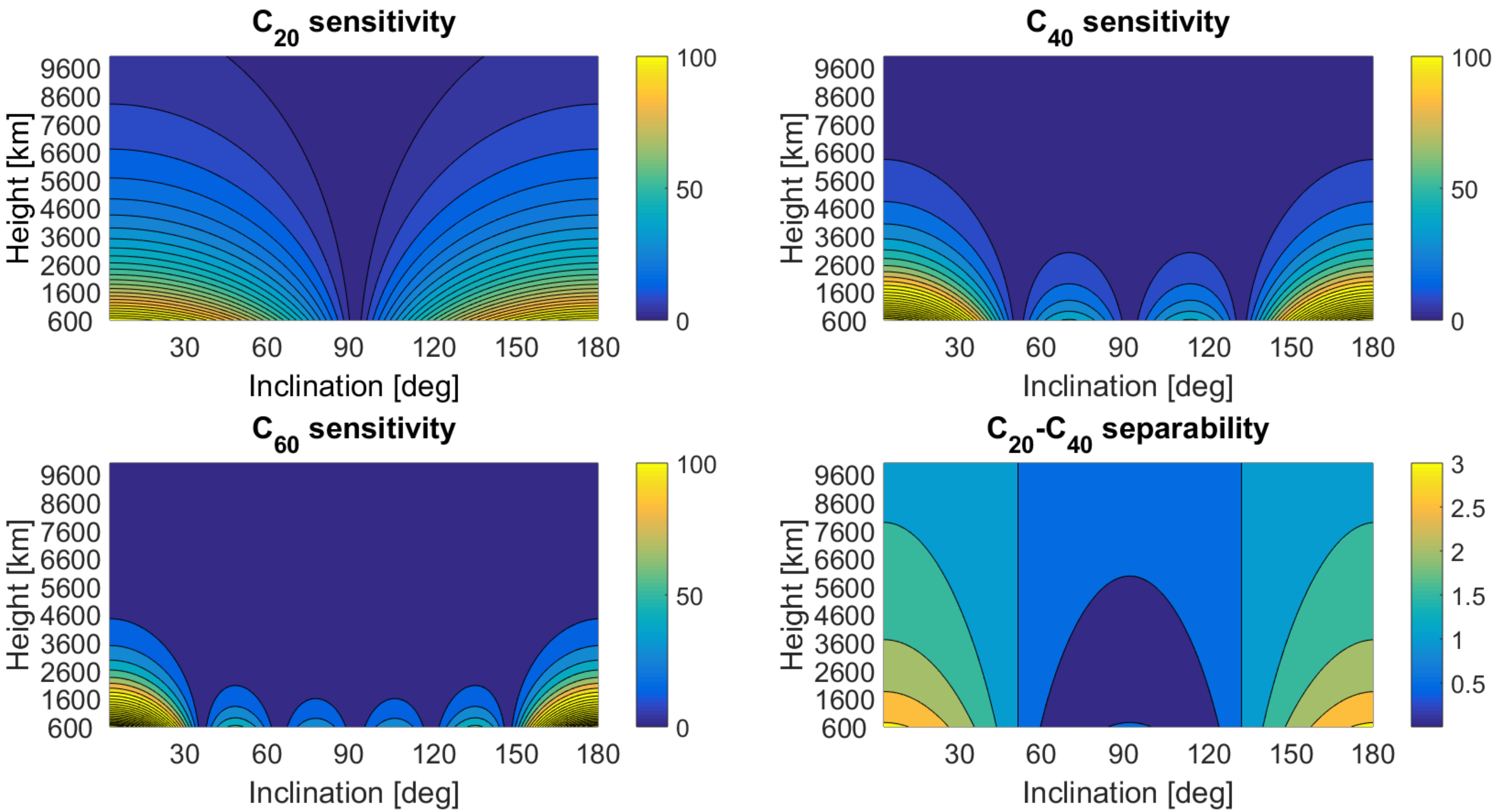
$$\frac{dF}{di} = \frac{3\cos(i)\sin(i)}{2}$$

$$G = \left(\frac{e^2}{(\sqrt{1-e^2}+1)^2}+1\right)^2\left(\frac{4e^2}{(\sqrt{1-e^2}+1)^2}+\frac{9e^4}{(\sqrt{1-e^2}+1)^4}+\ldots+1\right)$$

G=1 for low eccentricities

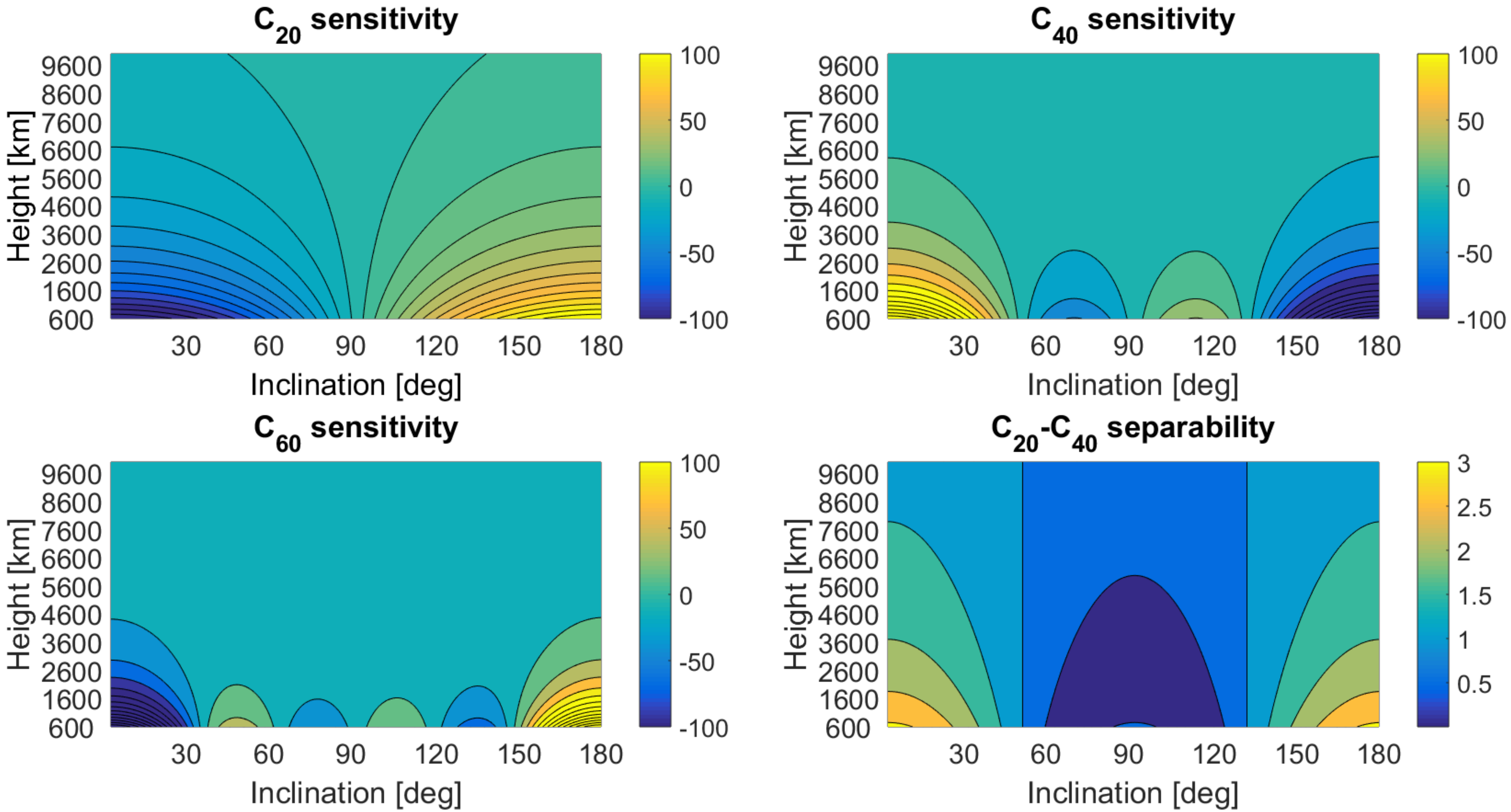
Example – C_{20}, C_{40}, C_{60}

Absolute differences



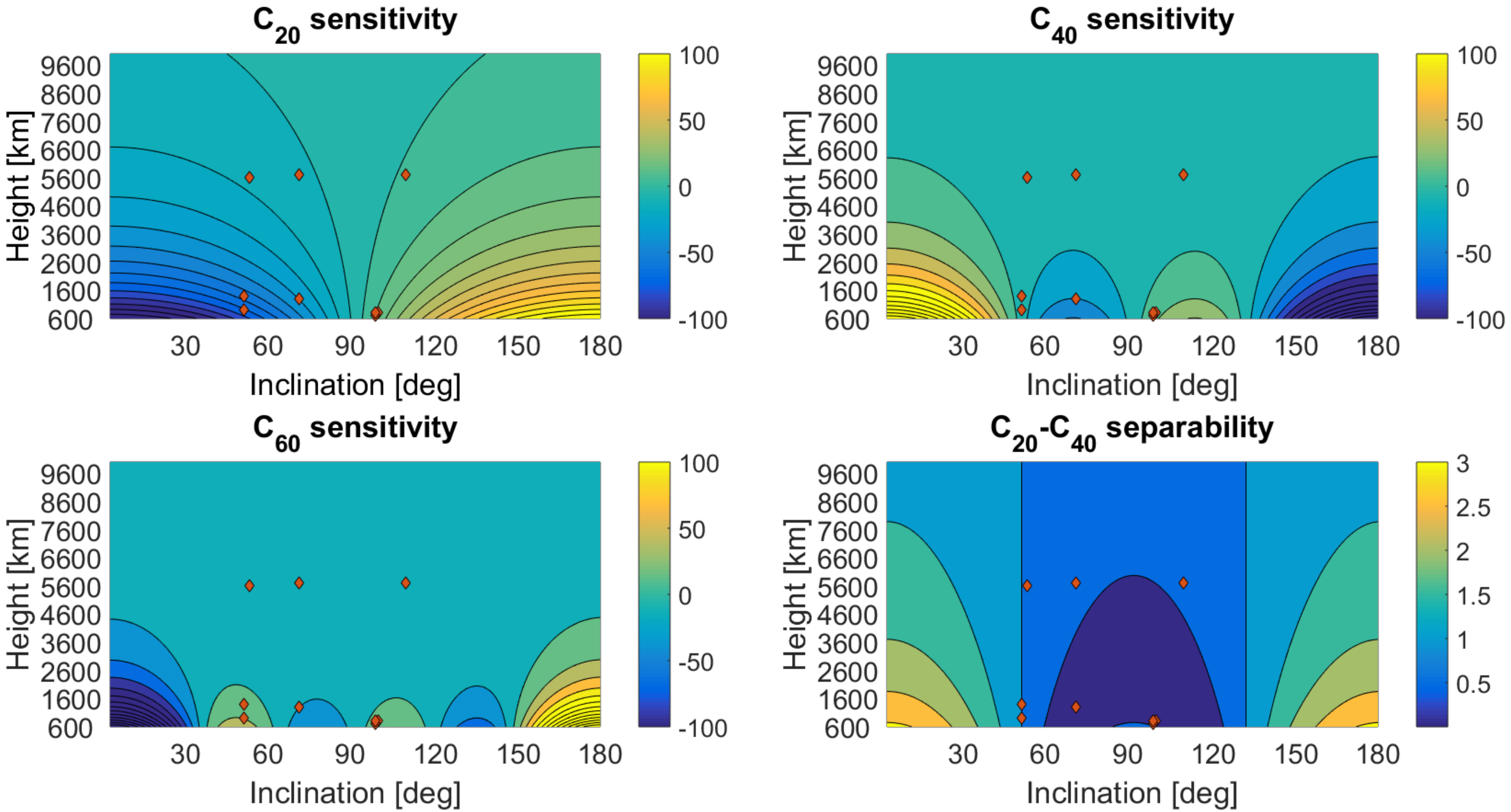
Example – C_{20}, C_{40}, C_{60}

Sensitivity with the direction of secular drift of ascending node



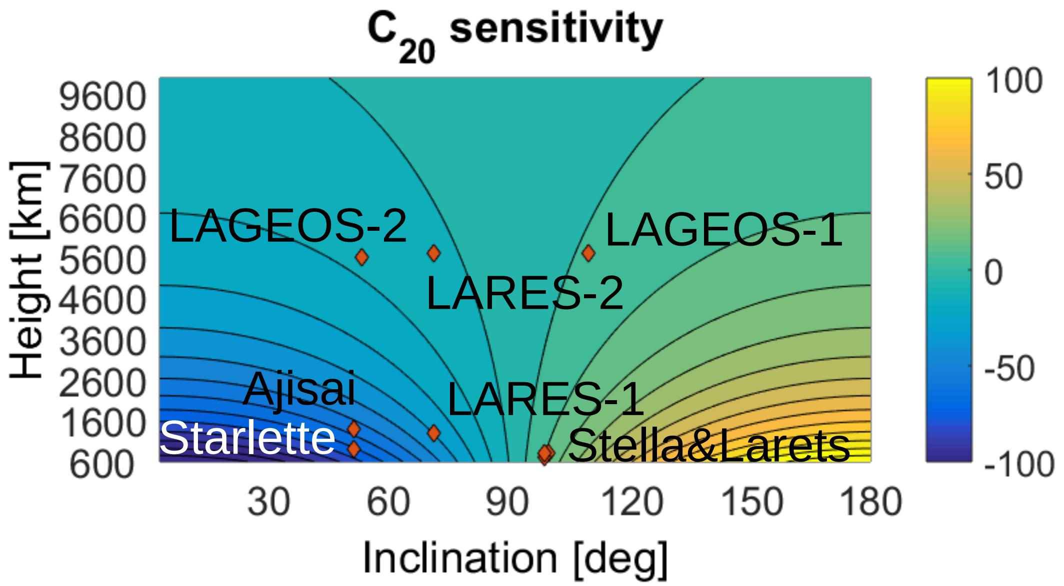
Example – C_{20}, C_{40}, C_{60}

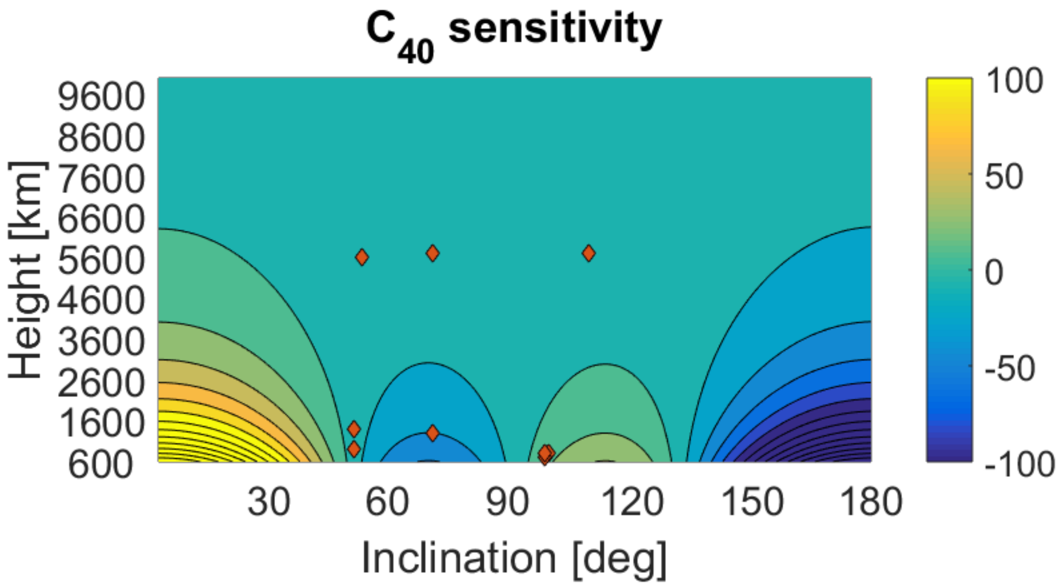
Sensitivity with the direction of secular drift of ascending node

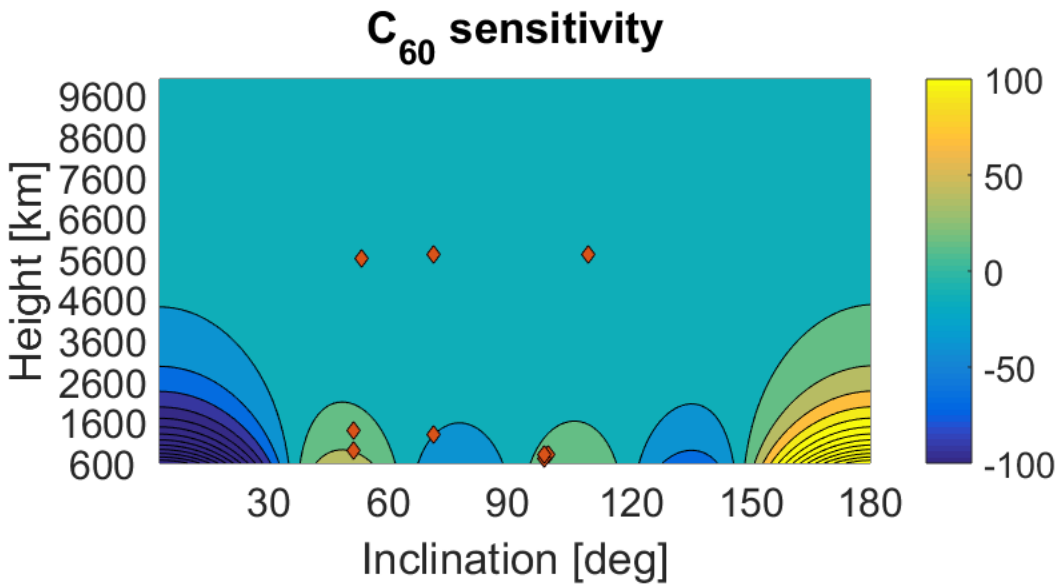


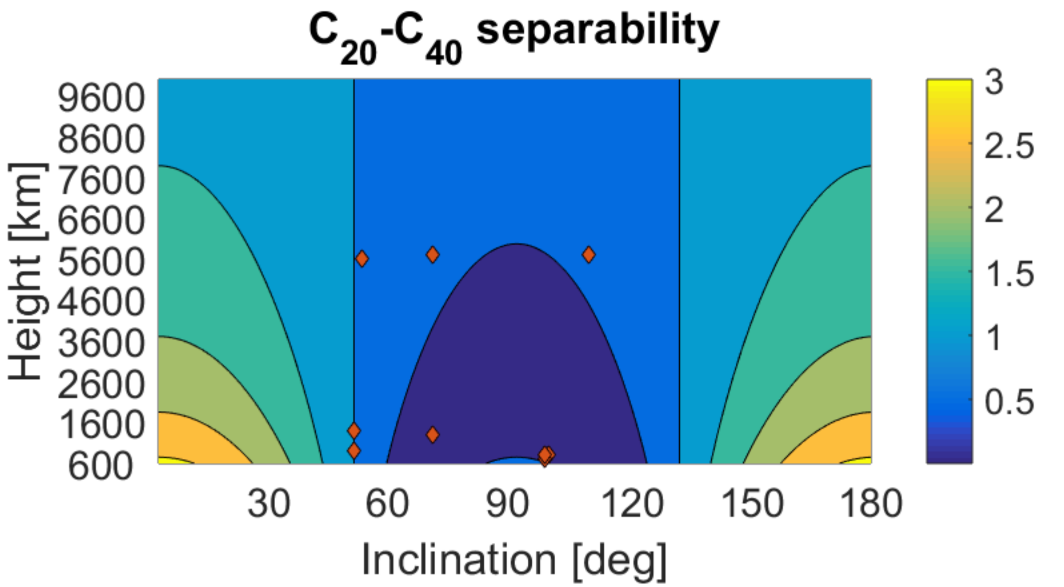
Example – C_{20}, C_{40}, C_{60}

$$Sp_{20-40} = \left| \frac{D_{20}}{D_{40}} - 1 \right|$$









Example – C_{30}

Recovery of **odd zonals** based on excitations of the eccentricity vector (**perigee**):

$$D_{30} = \frac{\sqrt{\frac{GM}{r^3}}}{2}\left(\frac{a_e}{r}\right)^3\sqrt{1-e^2}\,\frac{1}{e}\left(\left(G + e\frac{dG}{de}\right)F - \frac{e^2 G}{1-e^2}\,\frac{\cos(\mathrm{i})}{\sin(\mathrm{i})}\,\frac{dF}{di}\right)$$

n=3, m=0, p=1, q=-1

$$F = \frac{15\sin^3(\mathrm{i})}{16} - \frac{3\sin(\mathrm{i})}{4}$$

$$\frac{dF}{di} = \frac{45\cos(\mathrm{i})\sin(\mathrm{i})^2}{16} - \frac{3\cos(\mathrm{i})}{4}$$

$$G(e) = \frac{e^{20}}{\left(\sqrt{1-e^2}+1\right)^{20}} + \frac{e^{18}}{\left(\sqrt{1-e^2}+1\right)^{18}} + \frac{e^{16}}{\left(\sqrt{1-e^2}+1\right)^{16}} + \frac{e^{14}}{\left(\sqrt{1-e^2}+1\right)^{14}} + \frac{e^{12}}{\left(\sqrt{1-e^2}+1\right)^{12}} + \frac{e^{10}}{\left(\sqrt{1-e^2}+1\right)^{10}} + \frac{e^{8}}{\left(\sqrt{1-e^2}+1\right)^{8}} + \frac{e^{6}}{\left(\sqrt{1-e^2}+1\right)^{6}} + \frac{e^{4}}{\left(\sqrt{1-e^2}+1\right)^{4}} + \frac{e^{2}}{\left(\sqrt{1-e^2}+1\right)^{2}} + 1$$

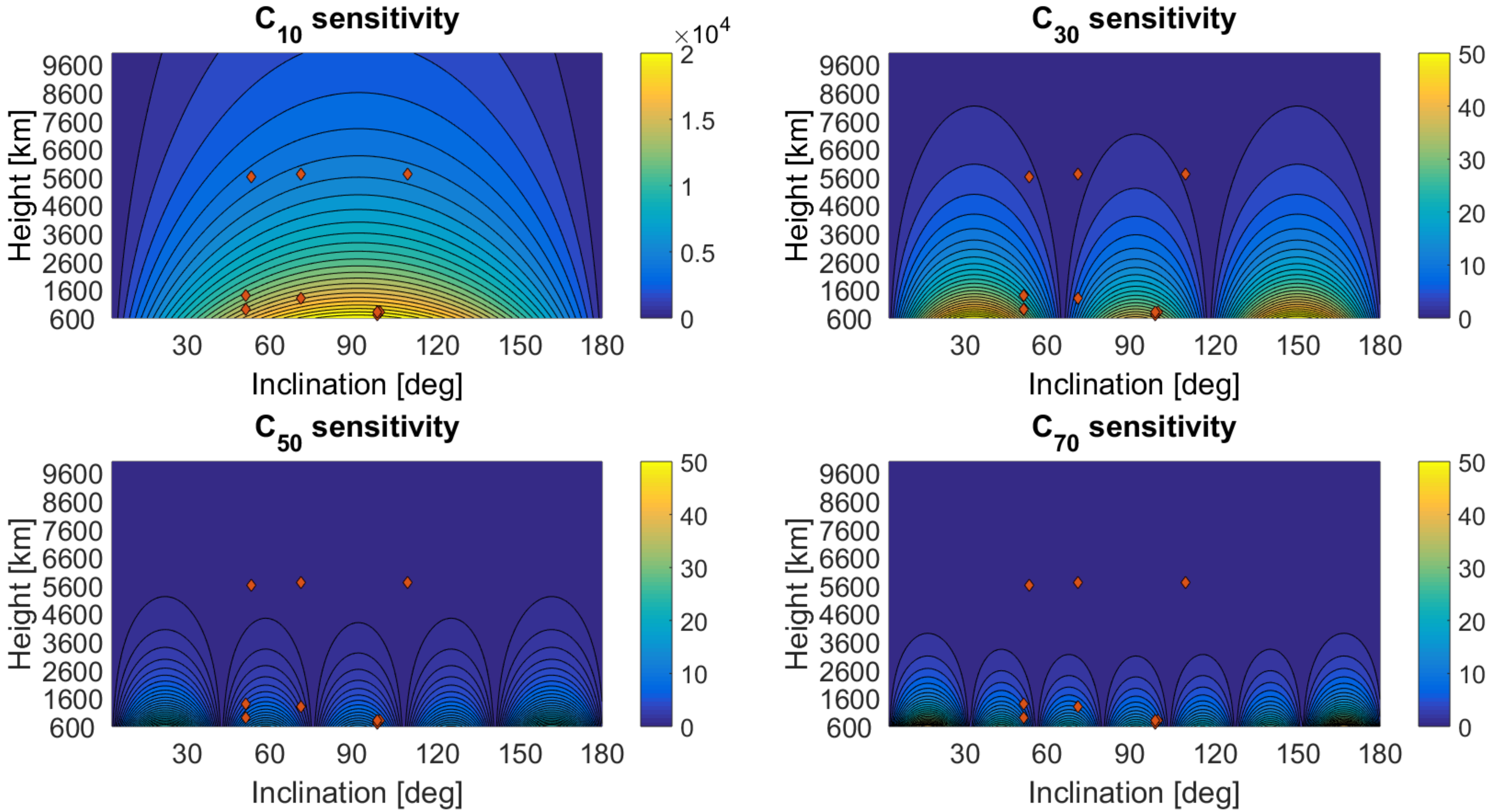
$$\frac{dG}{de} = 3\cdot\frac{e^2}{(\sqrt{1-e^2}+1)^2+1}\cdot\frac{d}{de}\left(\frac{1}{(\sqrt{1-e^2}+1)^2+1}\right)^2 + \frac{10e^9h(e)}{(\sqrt{1-e^2}+1)^{10}}$$

$$h(e) = 2\left(\frac{\sqrt{1-e^2}}{2}+\frac{1}{2}\right)^2 - \left(\frac{\sqrt{1-e^2}}{2}+\frac{1}{2}\right)^{3/2} + \frac{\left(\frac{\sqrt{1-e^2}}{2}+\frac{1}{2}\right)^4}{12} - \frac{\left(\frac{\sqrt{1-e^2}}{2}+\frac{1}{2}\right)^5}{120} - \frac{5\sqrt{1-e^2}}{2} + \frac{7}{2}$$

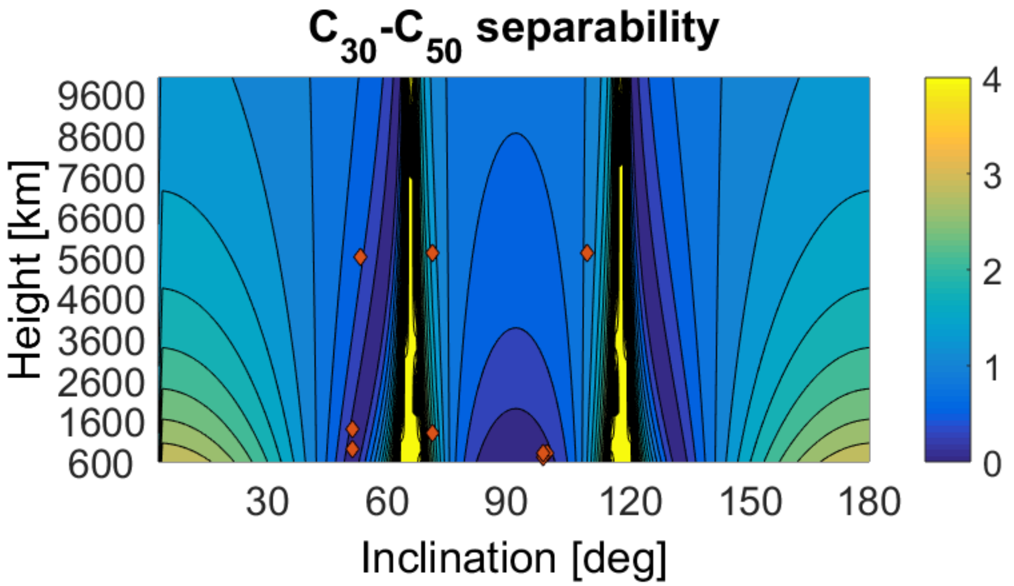
$$B = \frac{10e^9h(e)}{(\sqrt{1-e^2}+1)^{10}}$$

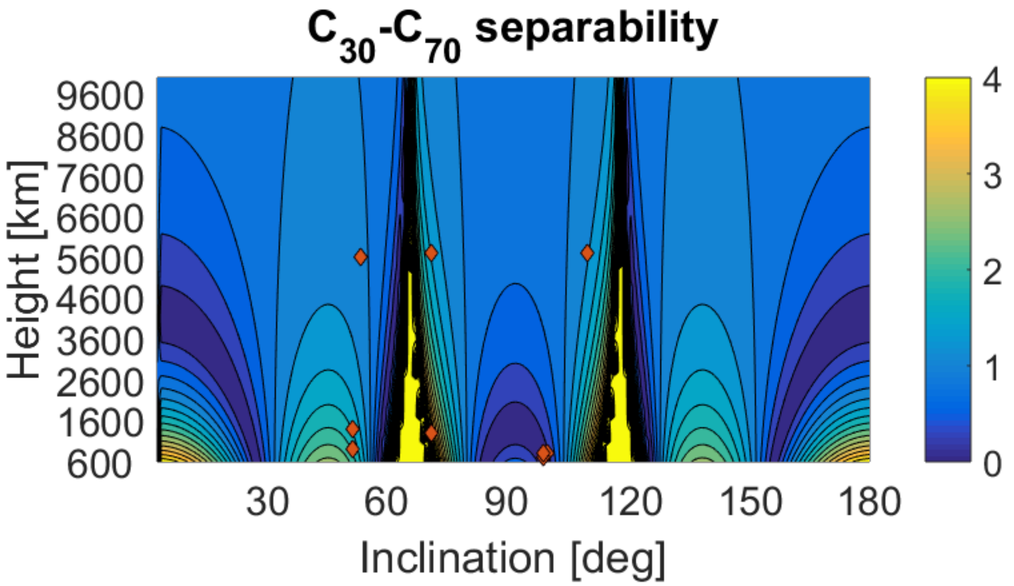
Example – $C_{10}, C_{30}, C_{50}, C_{70}$

For C_{10}, short-periodic perturbations are considered, for other odd-degree zonals – long-periodic perturbations -> different scales.



Example – C_{30} separability

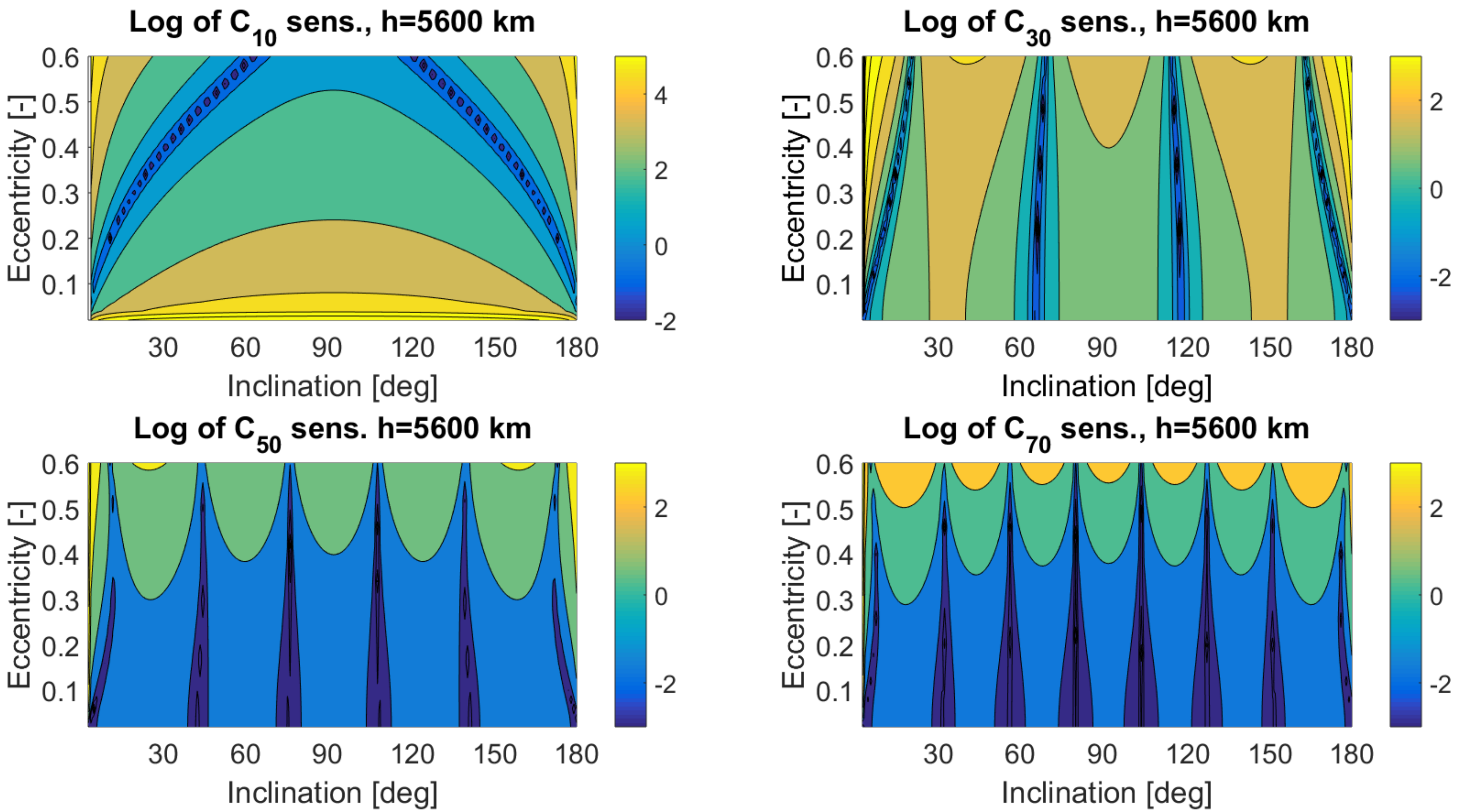




Best separability for critical inclinations: 63.4, 116.6 (no secular rates of perigee due to C_{20} & no long-periodic variations of perigee due to C_{30}).

Example – C_{30}

High-eccentricities are preferable for odd-harmonics, but not for geocenter (no long-periodic perigee perturbations occur for geocenter motion)



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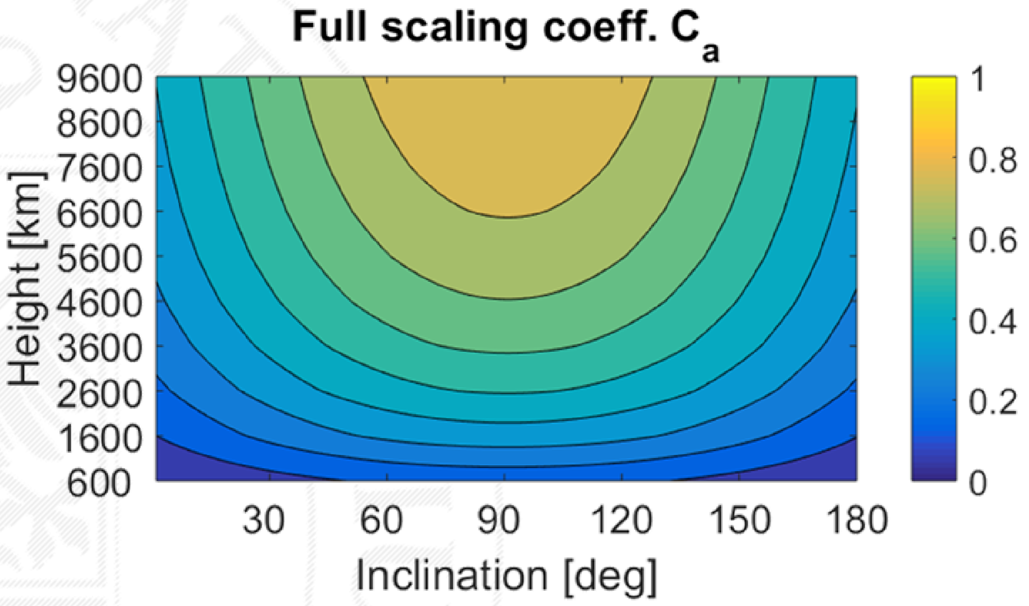
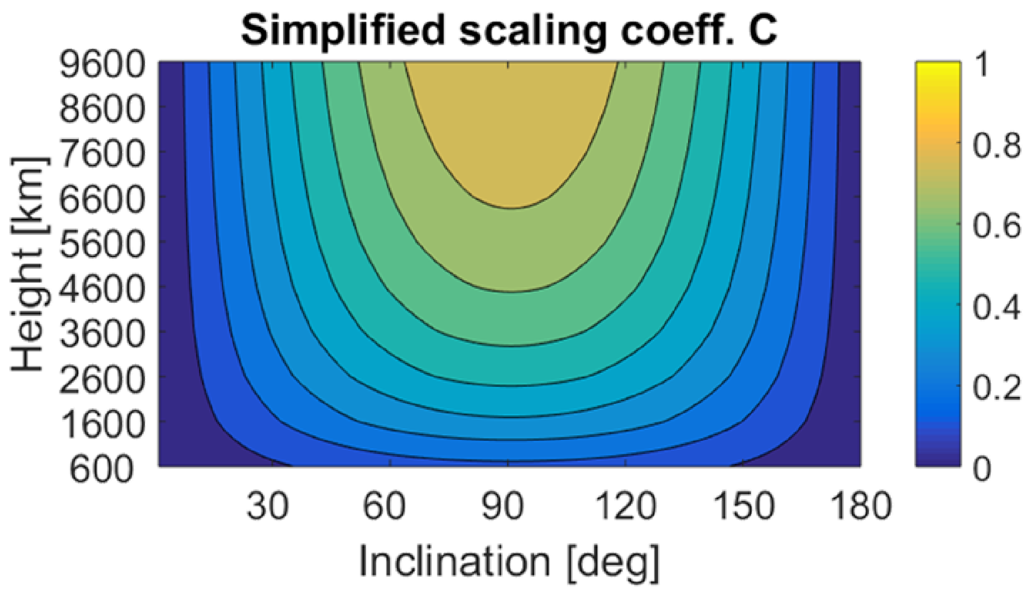
Satellite visibility function

Satellite visibility – Earth surface area from which a satellite can be tracked





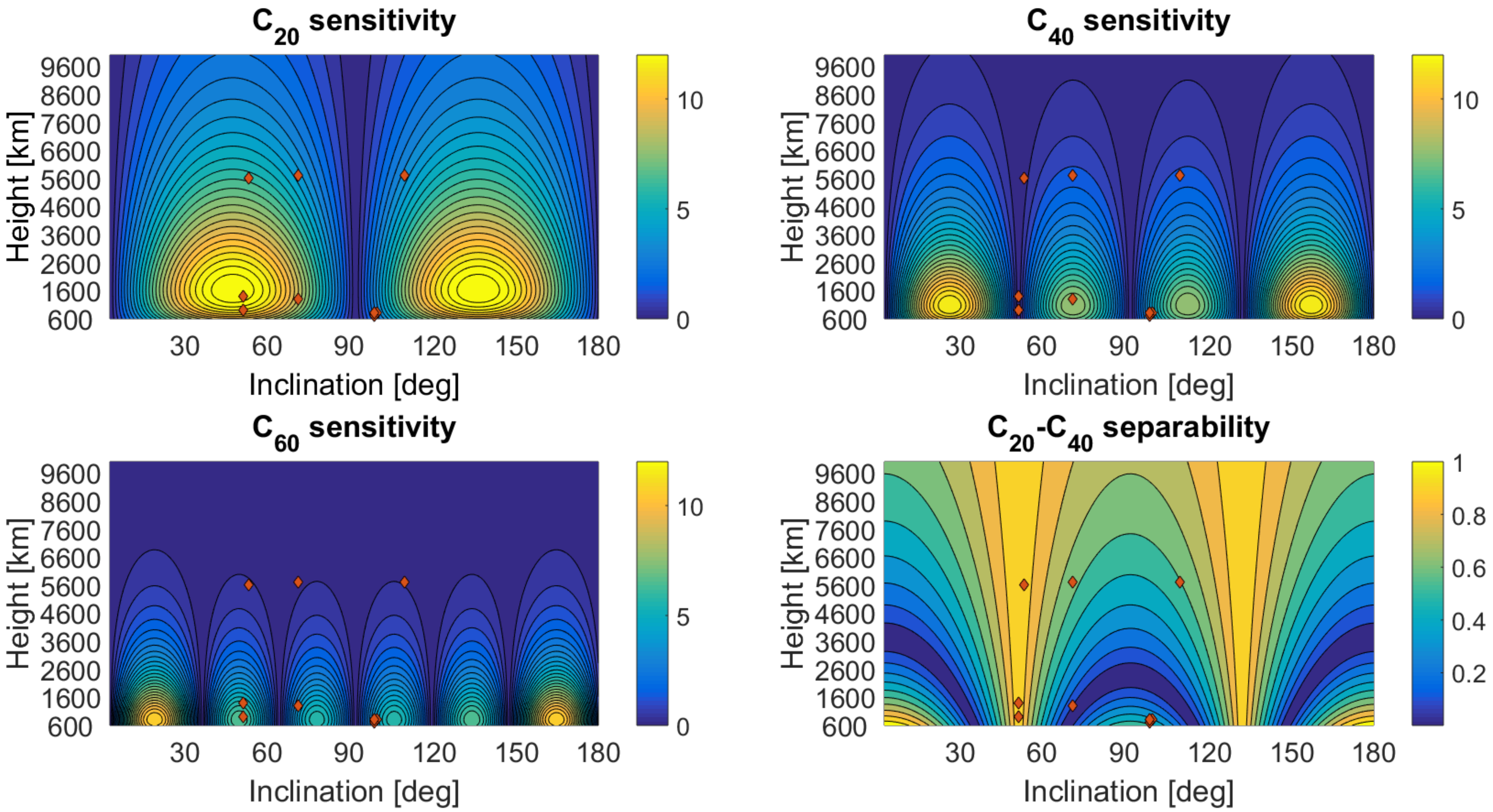
Function |sin i| excludes inclinations close to 0, however, for the gravity field recovery
- good tracking from SLR stations is needed
- good satellite coverage over different areas is preferable.

Therefore, i=0 is not used for geodetic satellites.

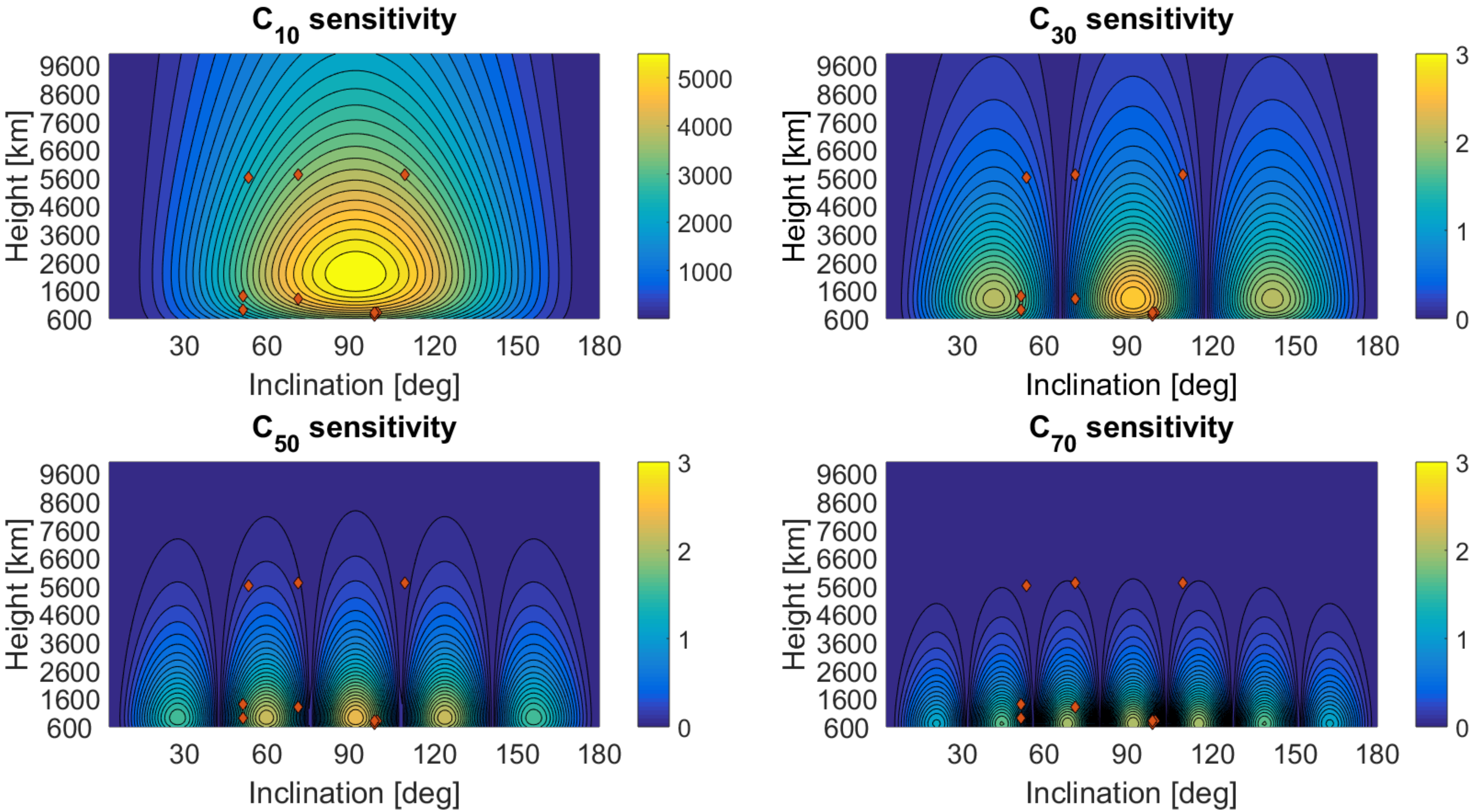
$$C_a = \left(1 - \frac{{a_e}^2}{(a_e+h)^2}\right) \frac{|\sin i| + \sqrt{1 - \frac{{a_e}^2}{(a_e+h)^2}}}{1 + \sqrt{1 - \frac{{a_e}^2}{(a_e+h)^2}}}$$

$$C = \left(1 - \frac{{a_e}^2}{(a_e+h)^2}\right) |\sin i|$$

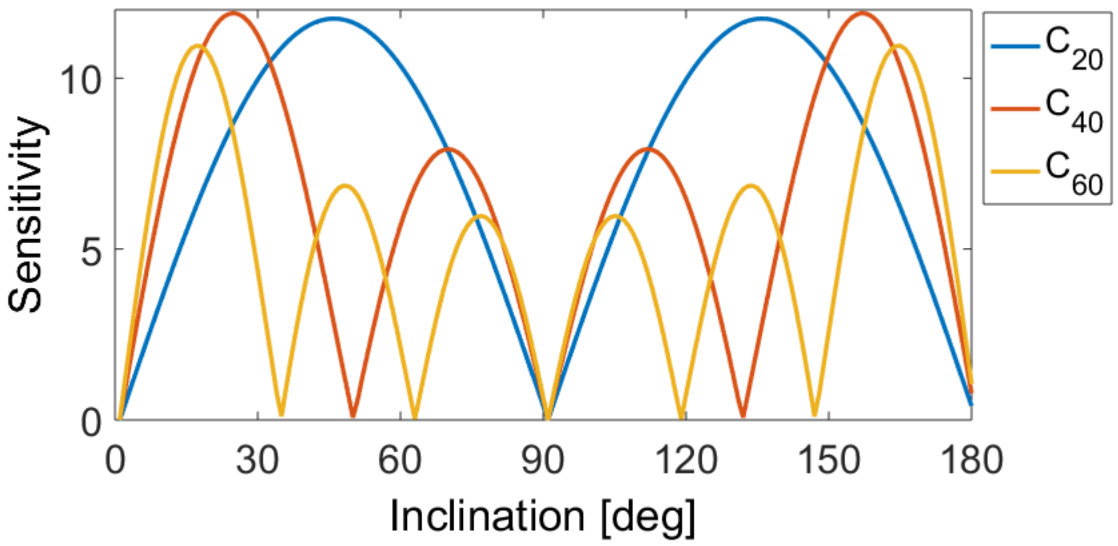
Best orbit for even zonal harmonics with satellite visibility factor

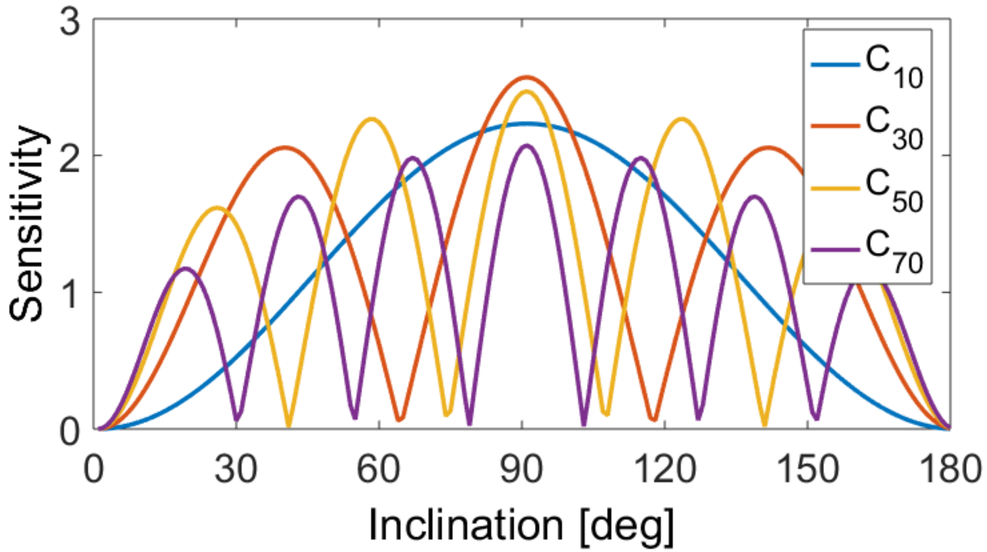


Best orbit for even zonal harmonics with satellite visibility factor



Best orbit inclinations for even and odd zonal harmonics





Despite reducing the impact for low inclination angles:

- Even degree harmonics prefer low inclinations (20-45 degrees) or (135-160 degrees for retrograte orbits- more revolutions in one day)
- Odd degree harmonics prefer inclination between 30 and 150 degrees. However, for inclinations 90, once-per-revolution empirical accelerations devoted to absorb the solar radiation pressure (SRP) or the constant SRP accelerations become strongly correlated with long-periodic variations.

Best orbit height for gravity field recovery

$$H_{opt}(n,r) \Leftrightarrow \max\left(\left(\frac{a_e}{r}\right)^n \sqrt{\frac{GM}{r^3}}\left(1-\frac{{a_e}^2}{r^2}\right)\right)$$

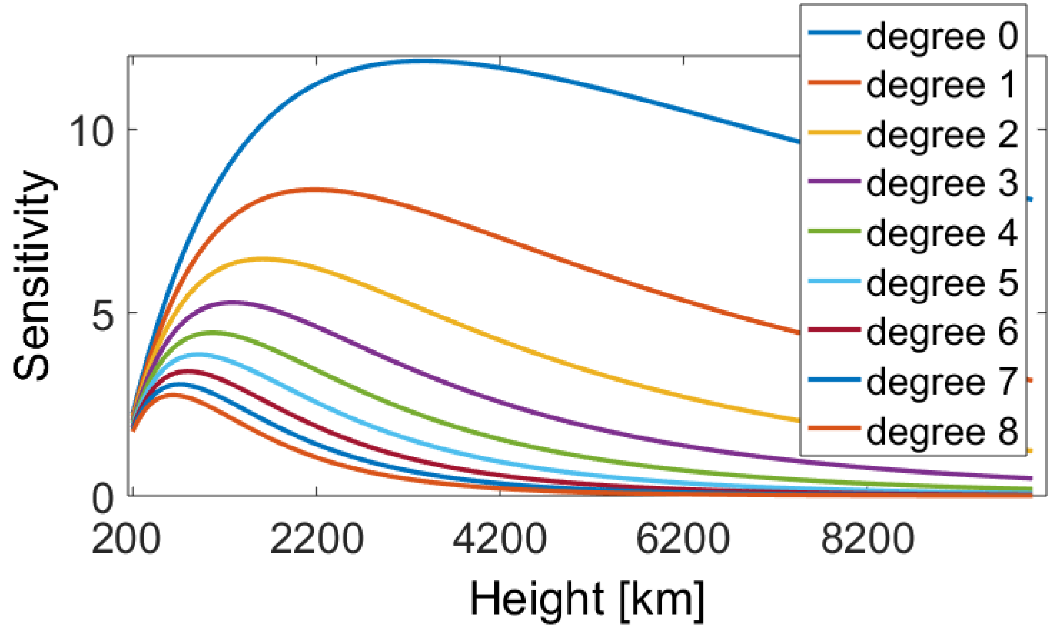
Sensitivity to gravity field (n^{th} – degree)

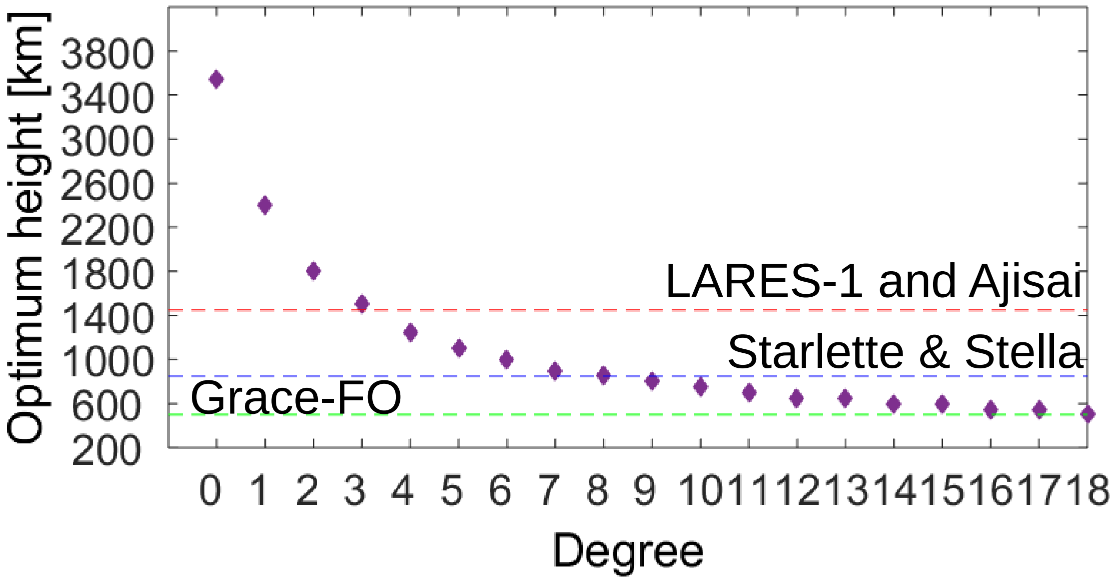
Mean motion – the number of possible passes to be registered by SLR stations

Earth surface area from which a satellite is visible

In reality, one must also consider other factors: (1) Earth rotation, (2) existing satellites, (3) correlations between SH parameters and their separability, (4) correlations with dynamical orbit parameters (SRP), (5) cloud conditions, (6) SLR station distribution, (7) performance of SLR stations, (8) range biases & other errors (e.g. CoM), (9) orbit modeling errors including the impact of the atmospheric drag.

Best orbit height for gravity field recovery





$$H_{opt}(n,r) \Leftrightarrow \max\left(\left(\frac{a_e}{r}\right)^n \sqrt{\frac{GM}{r^3}}\left(1-\frac{{a_e}^2}{r^2}\right)\right)$$

Optimum heights for vanishing inclinations:

- GM – 3500 km
- Geocenter – 2300 km
- Degree-2 – 1700 km
- Degree-3 – 1400 km
- Degree-4 – 1100 km

None of the existing satellites have optimum height for GM, geocenter, and degree-2!

Summary

Optimum heights of satellites for GM, geocenter, and degree-2 are in the areas where no geodetic satellites currently are; i.e, between 1700 and 3500 km

Best inclination angles for even-degree harmonics are in the range of 20-45 degrees for prograde or 135-160 degrees for retrograde orbit. Only Beacon-C is within this range (non-spherical satellite).

Best separability of odd-degree harmonics is for critical inclinations (63.4, 116.6 deg) or high eccentricities. C_{30} can be well determined from satellite at the inclination of 40 or 140 degrees at the height of 1400 km.

Retrograde orbits ensure more passes of the satellites than the prograde orbits in the same period.

Summary

To support future GRACE/MAGIC missions with C_{20} and C_{30} the best inclination would be 35-45 degrees or 135-145 degrees with the height about 1500-1700 km.

For superior geocenter recovery and determination of the gravitational constant, the best height would be 2300 – 3500 km. However, this study does not consider measurement errors, such as range biases, CoM, etc.

For higher-degree harmonics, e.g., (C_{50} and C_{70}), the best inclination would be 60 (120) or 63.4 (126.6) degrees with the height of 900 km. Eccentric orbits would be preferable for a better sensitivity to an odd-degree gravity field and separability with C_{30}.

Sośnica, K. (2024) Orbit design for a future geodetic satellite and gravity field recovery. *J Geod* **98**, 77 https://doi.org/10.1007/s00190-024-01884-9

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Thank you for your attention!

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